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Numerical Study of a Shape Design Problem in Elasticity by use of Auxiliary Domain Method

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Abstract. In this paper, we discuss numerical computation obtaining solution of a two-dimensional shape design problem in linear elasticity. Though this problem has been solved by using finite element method in [3], we present an alternative approach in order to use finite difference method. Auxiliary domain method is applied so that computation, especially at the boundary, can be more easier. One of the reasons why we use finite difference method is to avoid resetting of mesh in numerical computing while we change the domain which we consider. In that case, there are some ways with fixed mesh to solve the problem, for example, grid generation method. But these methods have some disadvantages such as losing linearity of the state equation and, obviously, become more complicated. Our technique provides to study numerical analysis of the problem as simple as possible.

1. Contact problem with friction

At first we will present steady state problem. Let a, b, c_o and c , where $a < b$ and $0 < c_o < c$, be given. We consider two-dimensional elastic body represented by

$$(1.1) \quad \Omega = \Omega(\alpha) = \{(x_1, x_2) \in R^2 \mid a < x_1 < b, \alpha(x_1) < x_2 < c\}$$

as shown in the figure 1.1.

$$(1.2) \quad \Gamma_c = \Gamma_c(\alpha) = \{(x_1, x_2) \in R^2 \mid a < x_1 < b, x_2 = \alpha(x_1)\}$$

contacts with the rigid body which is lower-half of the plane.

Here $\alpha \in U_{ad}$ and the set U_{ad} of admissible controls is defined by

$$(1.3) \quad U_{ad} = \left\{ \alpha \in C^{1,1}([a, b]) \mid \begin{array}{l} 0 \leq \alpha(x_1) \leq c_o, \\ |\alpha'(x_1)| \leq c_1 \quad \text{for } x_1 \in [a, b], \\ |\alpha''(x_1)| \leq c_2 \quad \text{a.e. in } (a, b), \\ \text{measure of } \Omega(\alpha) = c_3 \end{array} \right\}.$$

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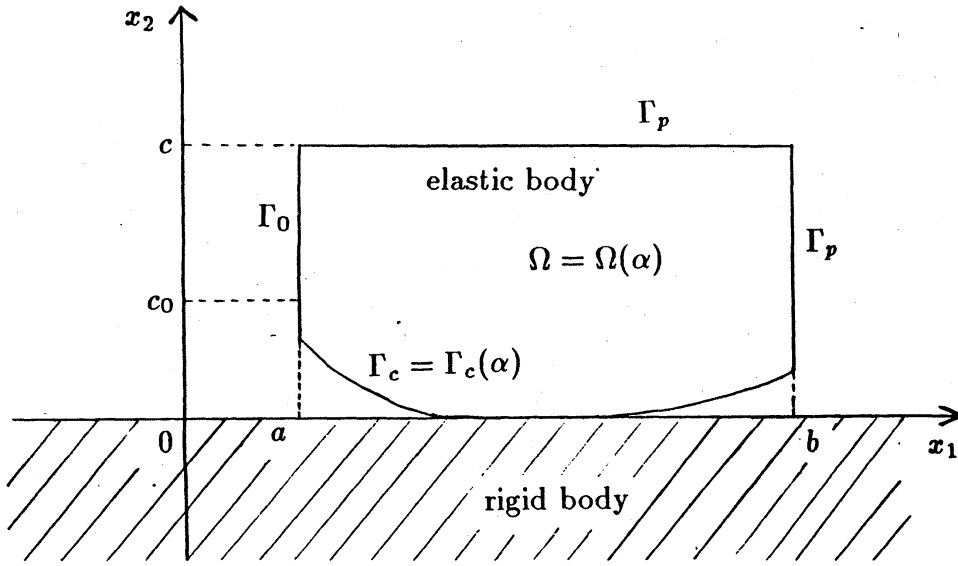


figure 1.1

The elastic body is subjected to forces and so deformation occurs. We denote the displacement vector by $u = (u_1(x_1, x_2), u_2(x_1, x_2))$. We assume that strain tensor

$$(1.4) \quad e_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2$$

and stress tensor $\tau_{ij}(u)$, $i, j = 1, 2$, satisfy the linearized Hooke's law

$$(1.5) \quad \tau_{ij}(u) = \lambda \delta_{ij} (e_{11}(u) + e_{22}(u)) + 2\mu e_{ij}(u), \quad i, j = 1, 2.$$

$\lambda \geq 0$ and $\mu > 0$ are Lamé's coefficients and constants. On the boundary of Ω , we define

$$(1.6) \quad T_i = T_i(u) = \sum_{j=1}^2 \tau_{ij}(u) n_j, \quad i = 1, 2,$$

where $n = (n_1, n_2)$ is outward unit normal vector.

Problem

Let force $f = (f_1, f_2)$ in Ω , stress $p = (p_1, p_2)$ on Γ_p and friction $g = \text{constant}$ on Γ_c be given. Then for some given $\alpha \in U_{ad}$, the problem is to find $u = u(\alpha)$ satisfying

$$(P) \quad \left\{ \begin{array}{l} -\sum_{j=1}^2 \frac{\partial \tau_{ij}(u)}{\partial x_j} = f_i, \quad i = 1, 2, \quad \text{in } \Omega \\ u_i = 0, \quad i = 1, 2, \quad \text{on } \Gamma_0 \\ T_i(u) = p_i, \quad i = 1, 2, \quad \text{on } \Gamma_p \\ u_2 + \alpha \geq 0, \quad T_2(u) \geq 0, \quad (u_2 + \alpha) T_2(u) = 0 \quad \text{on } \Gamma_c \\ |T_1(u)| \leq g, \quad (g - |T_1(u)|) u_1 = 0, \quad u_1 T_1(u) \leq 0 \quad \text{on } \Gamma_c \end{array} \right.$$

Differentiation which will be used throughout is taken in the sense of distribution.

In (P) , the condition $u_2 + \alpha \geq 0$ means that the elastic body can not penetrate into rigid body. On Γ_c , free boundaries will take place between touching parts where $u_2 + \alpha = 0$, $T_2(u) > 0$ and separating parts where $u_2 + \alpha > 0$, $T_2(u) = 0$. There may be another free boundaries between sliding parts where $|T_1(u)| = g$, $u_1 \neq 0$ and static parts where $|T_1(u)| < g$, $u_1 = 0$. See the figure 1.2 below. The condition $u_1 T_1(u) \leq 0$ states that, on sliding parts, u_1 and $T_1(u)$ are opposite.

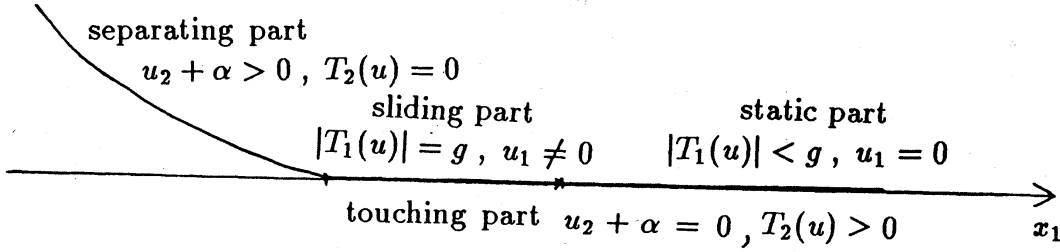


figure 1.2

2. Variational formulation

To express the problem (P) in weak form, we introduce the space

$$(2.1) \quad V = V(\alpha) = \left\{ v \in (H^1(\Omega(\alpha)))^2 \mid v_i = 0, \quad i = 1, 2, \quad \text{on } \Gamma_0 \right\},$$

nonempty closed convex set

$$(2.2) \quad K = K(\alpha) = \{ v \in V \mid v_2(x_1, \alpha(x_1)) + \alpha(x_1) \geq 0 \quad \text{a.e. in } (a, b) \},$$

continuous bilinear form

$$(2.3) \quad \begin{aligned} a(u, v) &= \sum_{i,j=1}^2 \int_{\Omega} \tau_{ij}(u) e_{ij}(v) dx, \quad u, v \in V, \\ &= \int_{\Omega} \left\{ \lambda \operatorname{div}(u) \operatorname{div}(v) + 2 \mu \sum_{i,j=1}^2 e_{ij}(u) e_{ij}(v) \right\} dx, \end{aligned}$$

continuous linear functional

$$(2.4) \quad \langle l, v \rangle = \sum_{i,j=1}^2 \int_{\Omega} f_i v_i dx + \sum_{i,j=1}^2 \int_{\Gamma_p} p_i v_i d\gamma, \quad v \in V$$

and continuous convex functional

$$(2.5) \quad j_c(v) = \int_{\Gamma_c} g |v_1| d\gamma, \quad v \in V.$$

It can be shown that (P) is equivalent to variational inequality

$$(VI) \quad \begin{cases} u = u(\alpha) \in K \text{ s.t.} \\ a(u, v - u) + j_c(v) - j_c(u) \geq \langle l, v - u \rangle, \text{ for all } v \in K. \end{cases}$$

For details, see [4]. Now the following classical result holds.

Theorem 2.1

Under the assumption of algebraic ellipticity condition for λ and μ , the solution of (VI) exists uniquely.

Since $a(.,.)$ is symmetric, we have equivalent minimization problem

$$(M) \quad u = u(\alpha) \in K \text{ s.t. } J(u) = \min_{v \in K} J(v),$$

where

$$(2.6) \quad J(v) = \frac{1}{2}a(v, v) - \langle l, v \rangle + j_c(v)$$

which is, in physical meaning, total potential energy.

3. Setting of the problem

For $\alpha \in U_{ad}$, we define cost functional by

$$(3.1) \quad E(\alpha) = J(u(\alpha)), \quad \text{where } u = u(\alpha) \text{ is the solution of (VI).}$$

Then shape design problem is to find

$$(SD) \quad \alpha^* \in U_{ad} \text{ s.t. } E(\alpha^*) = \min_{\alpha \in U_{ad}} E(\alpha).$$

We can explain that it is to look for an optimal domain on which displacement field u causes minimum of total potential energy. Existence of such domain is proved in the following.

Theorem 3.1 (J. Haslinger and P. Neittaanmaki) [3]

There exists at least one solution of the problem (SD).

4. Regularization and penalization

To approximate nondifferentiable functional j_c by a family of convex and differentiable functionals, we can regularize it as

$$(4.1) \quad j_{ce_r} = j_{ce_r}(v) = \int_{\Gamma_c} g \sqrt{v_1^2 + \varepsilon_r^2} d\gamma, \quad (\varepsilon_r > 0).$$

And to replace the fact that $u \in K$, a penalization operator $\beta : v \mapsto \beta(v)$ can be defined by

$$(4.2) \quad (\beta(v), w) = -\frac{1}{\varepsilon_p} \int_{\Gamma_c} (v_2 + \alpha)^- w_2 d\gamma, \quad w \in V, \quad (\varepsilon_p > 0).$$

Denote $\varepsilon = (\varepsilon_r, \varepsilon_p)$. Now we consider the following regularized and penalized variational equation

$$(VE) \quad \begin{cases} u_\varepsilon = u_\varepsilon(\alpha) \in V \text{ s.t.} \\ a(u_\varepsilon, v) + (j'_{ce_r}(u_\varepsilon) + \beta(u_\varepsilon), v) = \langle l, v \rangle, \text{ for all } v \in V. \end{cases}$$

Note that the operator $v \mapsto j'_{ce_r}(v)$ and β are monotone. Furthermore kernel of $\beta = K$ and β is lipschitz continuous. By using these facts, we get the following convergence result.[1][2]

Theorem 4.1

There exists unique solution u_ε of (VE) and $u_\varepsilon \rightarrow u$: solution of (VI) strongly in V as $\varepsilon = (\varepsilon_r, \varepsilon_p) \rightarrow (0, 0)$.

As before, (VE) is the same as the problem

$$(M_\varepsilon) \quad u_\varepsilon = u_\varepsilon(\alpha) \in V \text{ s.t. } J_\varepsilon(u_\varepsilon) = \min_{v \in V} J_\varepsilon(v),$$

where

$$(4.3) \quad J_\varepsilon(v) = \frac{1}{2}a(v, v) - \langle l, v \rangle + j_{ce_r}(v) + \frac{1}{2\varepsilon_p} \Psi(v)$$

and

$$(4.4) \quad \Psi(v) = \int_{\Gamma_c} [(v_2 + \alpha)^-]^2 d\gamma.$$

The problem becomes to find $u_\epsilon = (u_{\epsilon 1}, u_{\epsilon 2})$ satisfying

$$(P_\epsilon) \quad \left\{ \begin{array}{ll} -\sum_{j=1}^2 \frac{\partial \tau_{ij}(u_\epsilon)}{\partial x_j} = f_i, & i = 1, 2, \text{ in } \Omega \\ u_{\epsilon i} = 0, & i = 1, 2, \text{ on } \Gamma_0 \\ T_i(u_\epsilon) = p_i, & i = 1, 2, \text{ on } \Gamma_p \\ T_2(u_\epsilon) = \frac{1}{\epsilon_p} (u_{\epsilon 2} + \alpha)^- & \text{on } \Gamma_c \\ T_1(u_\epsilon) = -g \frac{u_{\epsilon 1}}{\sqrt{u_{\epsilon 1}^2 + \epsilon_r^2}} & \text{on } \Gamma_c. \end{array} \right.$$

One can notice that the boundary conditions on Γ_c become more simpler because of regularization and penalization. It will also be helpful in calculating the derivative of cost functional.

5. Auxiliary domain method

We construct an auxiliary domain $\tilde{\Omega}$ as shown in the following figure 5.1 and denote $\hat{\Omega} = (a, b) \times (\tilde{c}, c)$.

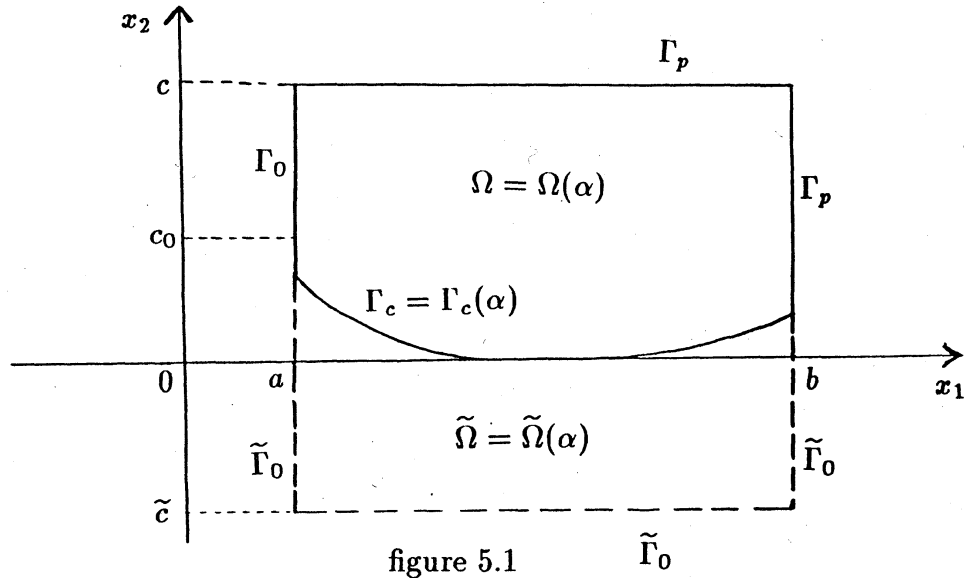


figure 5.1

Let us define again the space

$$(5.1) \quad \hat{V} = \left\{ v \in \left(H^1(\hat{\Omega}) \right)^2 \mid v_i = 0, \quad i = 1, 2, \quad \text{on } \Gamma_0 \cup \tilde{\Gamma}_0 \right\}.$$

Let $\epsilon_d > 0$. Our aim is to suppose the minimization problem, in place of (M_ϵ) ,

$$(\hat{M}_\epsilon) \quad \hat{u}_\epsilon = \hat{u}_\epsilon(\alpha) \in \hat{V} \quad \text{s.t.} \quad \hat{J}_\epsilon(\hat{u}_\epsilon) = \min_{v \in \hat{V}} \hat{J}_\epsilon(v),$$

where

$$(5.2) \quad \begin{aligned} \hat{J}_\varepsilon(v) = & \frac{1}{2}a(v, v) + \frac{1}{2} \varepsilon_d \tilde{a}(v, v) + \frac{1}{2} \int_{\tilde{\Omega}} |v|^2 dx \\ & - \langle l, v \rangle + j_{c\varepsilon_r}(v) + \frac{1}{2\varepsilon_p} \Psi(v). \end{aligned}$$

$\tilde{a}(.,.)$ is as in (2.3), but integration is based on the domain $\tilde{\Omega}$. Here $\varepsilon = (\varepsilon_r, \varepsilon_p, \varepsilon_d)$. We can explain (\hat{M}_ε) as

$$(\hat{P}_\varepsilon) \quad \left\{ \begin{array}{ll} -\sum_{j=1}^2 \frac{\partial \tau_{ij}(\hat{u}_\varepsilon)}{\partial x_j} = f_i, & i = 1, 2, \text{ in } \Omega \\ \hat{u}_{\varepsilon i} = 0, & i = 1, 2, \text{ on } \Gamma_0 \\ T_i(\hat{u}_\varepsilon) = p_i, & i = 1, 2, \text{ on } \Gamma_p \\ -\varepsilon_d \sum_{j=1}^2 \frac{\partial \tau_{ij}(\hat{u}_\varepsilon)}{\partial x_j} + \hat{u}_{\varepsilon i} = 0, & i = 1, 2, \text{ in } \tilde{\Omega} \\ \hat{u}_{\varepsilon i} = 0, & i = 1, 2, \text{ on } \tilde{\Gamma}_0 \\ T_2(\hat{u}_\varepsilon) + \varepsilon_d \tilde{T}_2(\hat{u}_\varepsilon) = \frac{1}{\varepsilon_p} (\hat{u}_{\varepsilon 2} + \alpha)^- & \text{on } \Gamma_c \\ T_1(\hat{u}_\varepsilon) + \varepsilon_d \tilde{T}_1(\hat{u}_\varepsilon) = -g \frac{\hat{u}_{\varepsilon 1}}{\sqrt{\hat{u}_{\varepsilon 1}^2 + \varepsilon_r^2}} & \text{on } \Gamma_c. \end{array} \right.$$

After all shape design problem to be considered is

$$(\widehat{SD}_\varepsilon) \quad \alpha_\varepsilon^* \in U_{ad} \text{ s.t. } \hat{E}_\varepsilon(\alpha_\varepsilon^*) = \min_{\alpha \in U_{ad}} \hat{E}_\varepsilon(\alpha),$$

and the cost functional to be optimized is

$$(5.3) \quad \hat{E}_\varepsilon(\alpha) = \hat{J}_\varepsilon(\hat{u}_\varepsilon(\alpha)) \quad , \quad \text{where } \hat{u}_\varepsilon = \hat{u}_\varepsilon(\alpha) \text{ is the solution of } (\hat{M}_\varepsilon).$$

6. Derivative of cost functional

We need to calculate Frechet derivative of $\hat{E}_\varepsilon(\alpha)$ to be used in optimizer. Let us state two lemmas which are useful.

Lemma 6.1

Let $C(\alpha)$ be smooth function which is defined in $\Omega(\alpha)$. Then

$$\begin{aligned} \frac{d}{d\alpha} \left[\int_{\Omega(\alpha)} C(\alpha) dx \right] \delta\alpha = & \int_{\Omega(\alpha)} \frac{\partial C(\alpha)}{\partial \alpha} \delta\alpha dx \\ & + \int_{\partial\Omega(\alpha)} C(\alpha) n \cdot \nu \delta\alpha d\gamma. \end{aligned}$$

Lemma 6.2

Under the same conditions as in Lemma 6.1, we have

$$\begin{aligned} \frac{d}{d\alpha} \left[\int_{\partial\Omega(\alpha)} C(\alpha) d\gamma \right] \delta\alpha &= \int_{\partial\Omega(\alpha)} \frac{\partial C(\alpha)}{\partial\alpha} \delta\alpha d\gamma \\ &+ \int_{\partial\Omega(\alpha)} \{ \nabla(C(\alpha)) \cdot n + H C(\alpha) \} n \cdot \nu \delta\alpha d\gamma. \end{aligned}$$

Here ∇ is gradient operator with respect to x , H is curvature of the curve $\partial\Omega(\alpha)$ and ν is to be taken as $(0, 1)$ or $(0, -1)$.

Let us denote $w = (w_1, w_2) = \left(\frac{\partial \hat{u}_{\epsilon 1}}{\partial \alpha}, \frac{\partial \hat{u}_{\epsilon 2}}{\partial \alpha} \right)$. With the help of Lemmas, we derive to obtain the followings.

$$\begin{aligned} (6.1) \quad & \frac{d}{d\alpha} \left[\frac{1}{2} \int_{\Omega} \sum_{i,j=1}^2 \tau_{ij}(\hat{u}_{\epsilon}) e_{ij}(\hat{u}_{\epsilon}) dx \right] \delta\alpha \\ &= - \int_{\Omega} \sum_{i,j=1}^2 \frac{\partial \tau_{ij}(\hat{u}_{\epsilon})}{\partial x_j} w_i dx + \int_{\Gamma_c} \sum_{i=1}^2 T_i(\hat{u}_{\epsilon}) w_i d\gamma \\ &+ \int_{\Gamma_p} \sum_{i=1}^2 T_i(\hat{u}_{\epsilon}) w_i d\gamma \\ &+ \int_{\Gamma_c} \frac{1}{2} \sum_{i,j=1}^2 \tau_{ij}(\hat{u}_{\epsilon}) e_{ij}(\hat{u}_{\epsilon}) n \cdot \nu \delta\alpha d\gamma \end{aligned}$$

$$\begin{aligned} (6.2) \quad & \frac{d}{d\alpha} \left[\frac{\varepsilon_d}{2} \int_{\tilde{\Omega}} \sum_{i,j=1}^2 \tau_{ij}(\hat{u}_{\epsilon}) e_{ij}(\hat{u}_{\epsilon}) dx \right] \delta\alpha \\ &= - \varepsilon_d \int_{\tilde{\Omega}} \sum_{i,j=1}^2 \frac{\partial \tau_{ij}(\hat{u}_{\epsilon})}{\partial x_j} w_i dx + \varepsilon_d \int_{\Gamma_c} \sum_{i=1}^2 \tilde{T}_i(\hat{u}_{\epsilon}) w_i d\gamma \\ &- \int_{\Gamma_c} \frac{\varepsilon_d}{2} \sum_{i,j=1}^2 \tilde{\tau}_{ij}(\hat{u}_{\epsilon}) \tilde{e}_{ij}(\hat{u}_{\epsilon}) n \cdot \nu \delta\alpha d\gamma \end{aligned}$$

$$\begin{aligned} (6.3) \quad & \frac{d}{d\alpha} \left[\frac{1}{2} \int_{\tilde{\Omega}} \sum_{i=1}^2 \hat{u}_{\epsilon i}^2 dx \right] \delta\alpha \\ &= \int_{\tilde{\Omega}} \sum_{i=1}^2 \hat{u}_{\epsilon i} w_i dx - \int_{\Gamma_c} \frac{1}{2} \sum_{i=1}^2 \hat{u}_{\epsilon i}^2 n \cdot \nu \delta\alpha d\gamma. \end{aligned}$$

$$\begin{aligned}
(6.4) \quad & \frac{d}{d\alpha} \left[- \int_{\Omega} \sum_{i=1}^2 f_i \hat{u}_{\epsilon i} dx \right] \delta\alpha \\
& = - \int_{\Omega} \sum_{i=1}^2 f_i w_i dx - \int_{\Gamma_c} \sum_{i=1}^2 f_i \hat{u}_{\epsilon i} n_{\nu} \delta\alpha d\gamma
\end{aligned}$$

$$(6.5) \quad \frac{d}{d\alpha} \left[- \int_{\Gamma_p} \sum_{i=1}^2 p_i \hat{u}_{\epsilon i} d\gamma \right] \delta\alpha = - \int_{\Gamma_p} \sum_{i=1}^2 p_i w_i d\gamma$$

$$\begin{aligned}
(6.6) \quad & \frac{d}{d\alpha} \left[\int_{\Gamma_c} g \sqrt{\hat{u}_{\epsilon 1}^2 + \epsilon_r^2} d\gamma \right] \delta\alpha \\
& = \int_{\Gamma_c} g \frac{\hat{u}_{\epsilon 1}}{\sqrt{\hat{u}_{\epsilon 1}^2 + \epsilon_r^2}} w_1 d\gamma \\
& \quad + \int_{\Gamma_c} g \left\{ \nabla (\sqrt{\hat{u}_{\epsilon 1}^2 + \epsilon_r^2}) \cdot n + H \sqrt{\hat{u}_{\epsilon 1}^2 + \epsilon_r^2} \right\} n_{\nu} \delta\alpha d\gamma
\end{aligned}$$

$$\begin{aligned}
(6.7) \quad & \frac{d}{d\alpha} \left[\frac{1}{2\epsilon_p} \int_{\Gamma_c} [(\hat{u}_{\epsilon 2} + \alpha)^-]^2 d\gamma \right] \delta\alpha \\
& = - \frac{1}{\epsilon_p} \int_{\Gamma_c} (\hat{u}_{\epsilon 2} + \alpha)^- w_2 d\gamma \\
& \quad + \frac{1}{2\epsilon_p} \int_{\Gamma_c} \left\{ \nabla ([(\hat{u}_{\epsilon 2} + \alpha)^-]^2) \cdot n \right. \\
& \quad \left. + H [(\hat{u}_{\epsilon 2} + \alpha)^-]^2 \right\} n_{\nu} \delta\alpha d\gamma
\end{aligned}$$

(6.1) ~ (6.7) and equations in (\hat{P}_{ϵ}) give the final result :

$$\begin{aligned}
(6.8) \quad & \frac{d}{d\alpha} [\hat{E}_{\epsilon}(\alpha)] \delta\alpha \\
& = \int_{\Gamma_c} \left[\frac{1}{2} \sum_{i,j=1}^2 \{ \tau_{ij}(\hat{u}_{\epsilon}) e_{ij}(\hat{u}_{\epsilon}) - \epsilon_d \tilde{\tau}_{ij}(\hat{u}_{\epsilon}) \tilde{e}_{ij}(\hat{u}_{\epsilon}) \} \right. \\
& \quad - \sum_{i=1}^2 \left\{ \frac{1}{2} \hat{u}_{\epsilon i}^2 + f_i \hat{u}_{\epsilon i} \right\} \\
& \quad + \nabla (g \sqrt{\hat{u}_{\epsilon 1}^2 + \epsilon_r^2} + \frac{1}{2\epsilon_p} [(\hat{u}_{\epsilon 2} + \alpha)^-]^2) \cdot n \\
& \quad \left. + H \left\{ g \sqrt{\hat{u}_{\epsilon 1}^2 + \epsilon_r^2} + \frac{1}{2\epsilon_p} [(\hat{u}_{\epsilon 2} + \alpha)^-]^2 \right\} \right] n_{\nu} \delta\alpha d\gamma.
\end{aligned}$$

Since w is cancelled, adjoint equation is not necessary to be considered. We will denote the right side of (6.8) by $\int_{\Gamma_c} \phi \delta \alpha d\gamma$.

In actual computation, we want to work out with the admissible set

$$(6.9) \quad W_{ad} = \left\{ \alpha \in C^{1,1}([a,b]) \mid \begin{aligned} &0 \leq \alpha(x_1) \leq c_0, \\ &|\alpha'(x_1)| \leq c_1 \quad \text{for } x_1 \in [a,b], \\ &|\alpha''(x_1)| \leq c_2 \quad \text{a.e. in } (a,b) \end{aligned} \right\}$$

from which restriction for measure of $\Omega(\alpha)$ is excluded. But it is unavoidable to retain that condition. Therefore we try to include it again in cost functional as

$$(6.10) \quad \widehat{F}_\epsilon(\alpha) = \widehat{E}_\epsilon(\alpha) + k |c_3 - m(\Omega)|^2,$$

where k is a positive constant and $m(\Omega)$ is notation for measure of $(\Omega(\alpha))$.

At last it becomes to study numerically the shape design problem

$$(\widehat{\mathcal{SD}}_\epsilon) \quad \alpha_\epsilon^* \in W_{ad} \quad \text{s.t.} \quad \widehat{F}_\epsilon(\alpha_\epsilon^*) = \min_{\alpha \in W_{ad}} \widehat{F}_\epsilon(\alpha).$$

It can easily be seen that

$$(6.11) \quad \begin{aligned} &\frac{d}{d\alpha} [\widehat{F}_\epsilon(\alpha)] \delta \alpha \\ &= \int_{\Gamma_c} \phi \delta \alpha d\gamma + 2k (c_3 - m(\Omega)) \int_{\Gamma_c} \frac{1}{\sqrt{1 + (\alpha'(x_1))^2}} \delta \alpha d\gamma. \end{aligned}$$

For certain α , we can write the right side of (6.11) like as $\int_{\Gamma_c} \Phi \delta \alpha d\gamma$.

7. Optimizing algorithm

Numerical computations are carried out as stated below.

Step 0 : Initial $\alpha \in W_{ad}$ is given.

Step 1 : Solve (\widehat{M}_ϵ) .

Step 2 : Compute Φ .

Step 3 : Take $\delta \alpha = -\rho \Phi$ ($\rho > 0$).

Step 4 : Set $\alpha \leftarrow \alpha + \delta \alpha$.

Go to Step 1.

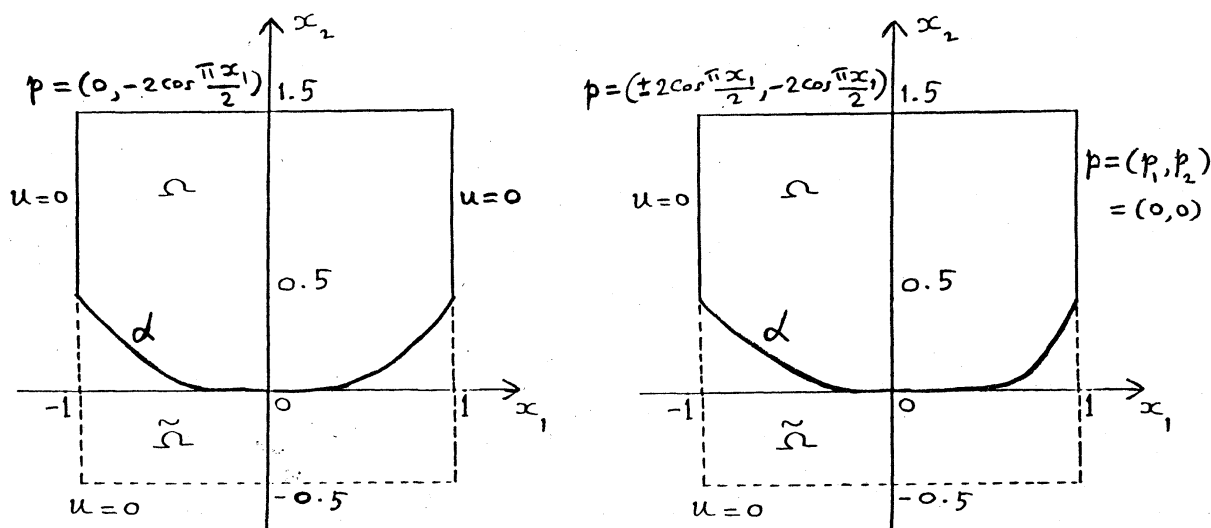
Relaxation method is used to solve (\widehat{M}_ϵ) . To optimize \widehat{F}_ϵ , according to Pironneau's method, we take $\delta\alpha$ as $\delta\alpha = -\rho \Phi$ ($\rho > 0$) so that $\frac{d}{d\alpha}[\widehat{F}_\epsilon(\alpha)] \delta\alpha \leq 0$. Suitable ρ had to be selected in order to that $\alpha + \delta\alpha$ lies in the admissible set W_{ad} .

8. Numerical examples

We take $f = (f_1, f_2) = (0, 0)$ for simplicity and use the data :

$$\begin{aligned} a = -1.0, b = 1.0, c = 1.5, \tilde{c} = -0.5 & \quad \text{mesh size } h = \frac{1}{8}, \frac{1}{16} \\ c_0 = 0.5, c_1 = 0.5, c_2 = 4.0, c_3 = 2.75 & \\ \lambda = 1.0, \mu = 1.0, g = 0.001 & \quad \epsilon_r, \epsilon_d = 2h^2, h^2, \frac{2h^2}{2} \\ k = 8.0 & \quad \epsilon_p = h, \frac{h}{2}, \frac{2h}{5} \end{aligned}$$

For $p = (p_1, p_2)$, see the following figures.



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